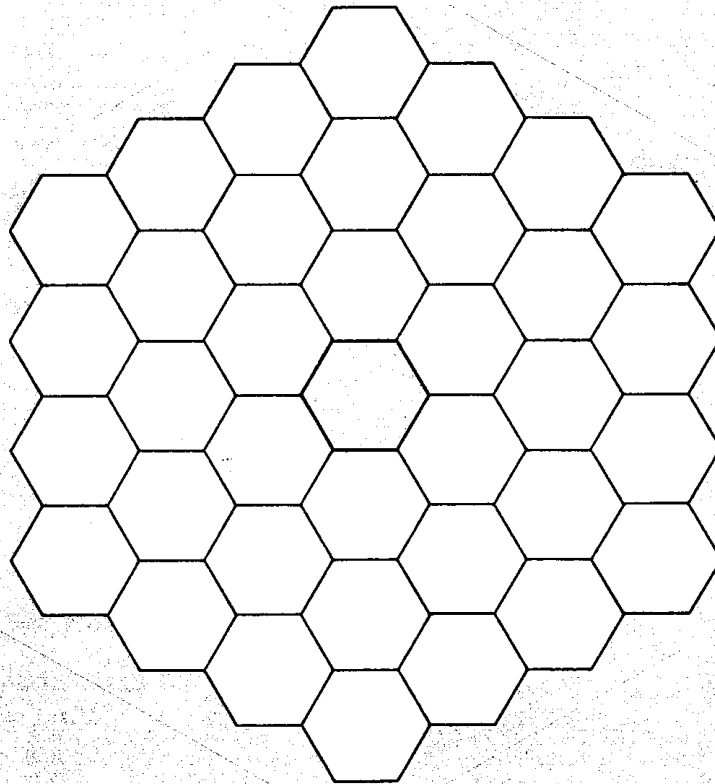


# The Off-Axis Expansion of Conic Surfaces

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## *ABSTRACT*

We derive a Taylor Series expansion for a general axisymmetric conic surface about an arbitrary point on the surface. The series is explicitly evaluated through 4th order. Such expansions are useful in evaluating the optical properties of segment surfaces used in the construction of segmented mirror telescopes. We show that the 4th order expansion is an adequate approximation for an extremely wide range of segment types.

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## 1. Introduction

In order to evaluate or fabricate an optical surface, it is first necessary to define mathematically the shape of that surface. For the case of axisymmetric conics, the formulas are well known, but when one wishes to describe such a surface in a coordinate system not aligned with the axis of symmetry, analytic expressions are much more complex, and not readily available. In addition, when one wishes to assemble an axisymmetric optical surface from segments, it is desirable to have convenient expressions for the segment surface in a coordinate system that is centered on the segment rather than centered on the optical axis. In this report we will determine the coefficients of a Taylor series expansion representing a general conic in such a coordinate system.

## 2. Axisymmetric Conics

A general expression for a conic surface of revolution is given by

$$Z(X, Y) = \frac{1}{K+1} \left\{ k - [k^2 - (K+1)S^2]^{1/2} \right\} \quad (1)$$
$$= \frac{S^2}{2k} + (K+1) \frac{S^4}{8k^3} + (K+1)^2 \frac{S^6}{16k^5} + \frac{5}{128} (K+1)^3 \frac{S^8}{k^7} + \dots$$

$K$  = conic constant ( $= -e^2$ ,  $e$  = eccentricity)

$k$  = radius of curvature

$S = (X^2 + Y^2)^{1/2}$

The geometry is indicated in Figure 1. For various values of  $K$ , this reduces to well known expressions

$K < -1$  Hyperboloid of revolution

$K = -1$  Paraboloid

$-1 < K < 0$  Prolate ellipsoid

$K = 0$  Sphere

$K > 0$  Oblate ellipsoid

The paraboloid expression can be found by taking the limit of equation (1) as  $K \rightarrow -1$ , to obtain

$$Z_{\text{paraboloid}} = \frac{S^2}{2k} \quad (2)$$

For a sphere, equation (1) reduces to

$$Z_{sphere} = k - (k^2 - S^2)^{1/2} \quad (3)$$

or

$$Z_{sphere} = \frac{S^2}{2k} + \frac{S^4}{8k^3} + \frac{S^6}{16k^5} + \frac{5}{128} \frac{S^8}{k^7} + \dots$$

### 3. Transformation to Off-axis Coordinates

We now consider the description of the conic as viewed from a coordinate system that is tangent to the surface at a general point away from the axis of symmetry. Figure 1 shows the geometry. We define  $R$  as the distance from the rotation axis to the new coordinate center; thus  $R = (X_0^2 + Y_0^2)^{1/2}$ . Without loss of generality we choose the new coordinate center on the  $Y$  axis.

We are interested in describing the surface of a segment of radius  $a$  in terms of the local coordinates  $x, y$ . We use the dimensionless polar variables  $\rho = (x^2 + y^2)^{1/2}/a$ ,  $\theta = \tan^{-1}(y/x)$ . A convenient general form for the description of the surface is

$$z(\rho, \theta) = \sum_{m,n} \alpha_{mn} \rho^m \cos n\theta + \beta_{mn} \rho^m \sin n\theta \quad m \geq n \geq 0, m - n \text{ even.} \quad (4)$$

By suitable selection of the coordinates, we can set  $\beta_{mn} = 0$ . Thus our objective is to derive expressions for the  $\alpha_{mn}$ . The global coordinates  $(X, Y)$  are related to the local coordinates  $(x, y)$  by a rotation  $\phi_0$  and a translation  $Z_0$ .

$$\tan \phi_0 = \left. \frac{\partial Z}{\partial X} \right|_{X=R} = \frac{R}{[k^2 - (K+1)R^2]^{1/2}} \quad (5)$$

$$Z_0 = Z(X=R, Y=0) = \frac{1}{K+1} (k - [k^2 - (K+1)R^2]^{1/2})$$

$$X = x \cos \phi_0 - z \sin \phi_0 + R \quad (6a)$$

$$Y = y \quad (6b)$$

$$Z = x \sin \phi_0 + z \cos \phi_0 + Z_0 \quad (6c)$$

$$x = (X - R) \cos \phi_0 + (Z - Z_0) \sin \phi_0 \quad (6d)$$

$$y = Y \quad (6e)$$

$$z = -(X - R) \sin \phi_0 + (Z - Z_0) \cos \phi_0 \quad (6f)$$

For compactness, we introduce the dimensionless variables

$$u = \frac{x}{k}, \quad v = \frac{y}{k}, \quad w = \frac{z}{k}, \quad \epsilon = \frac{R}{k}, \quad W = \frac{Z(X, Y)}{k} \quad (7a)$$

and the quantities

$$s \equiv \sin \phi_0 = \frac{R}{[k^2 - KR^2]^{1/2}} = \frac{\epsilon}{[1 - K\epsilon^2]^{1/2}}$$

$$c \equiv \cos \phi_0 = \left[ \frac{k^2 - (K+1)R^2}{k^2 - KR^2} \right]^{1/2} = \left[ \frac{1 - L\epsilon^2}{1 - K\epsilon^2} \right]^{1/2} \quad (7b)$$

$$L \equiv K + 1$$

$$W_0 \equiv Z_0/k \equiv (1/L)(1 - c\epsilon/s)$$

Now, using the variables in equation (7), we can rewrite equation (6c), evaluated at a point  $(X, Y)$  on the conic as

$$(1/L)(1 - [1 - L([uc - ws + \epsilon]^2 + v^2)]^{1/2}) = us + wc + W_0 \quad (8)$$

We wish to solve this equation for  $w(u, v)$ . First multiplying by  $L$ , subtracting 1, and squaring gives

$$1 - L([uc - ws + \epsilon]^2 + v^2) = (1 - L[wc + us + W_0])^2$$

Multiplying out the terms and collecting the coefficients of  $w$  yields

$$w^2[LC^2 + s^2] + 2w[c(LW_0 - 1) - s\epsilon + (L-1)scu] + [(Ls^2 + c^2)u^2 + v^2 + 2(s(LW_0 - 1) + c\epsilon)u + (LW_0^2 - 2W_0 + \epsilon^2)] = 0 \quad (9)$$

Using the definitions in Equation 7 one can show that each of the last two sums in parenthesis vanishes. To further simplify Equation 9 we define the following quantities

$$\begin{aligned} f &\equiv (s/\epsilon)^2 g \\ g &\equiv -1/(LC^2 + s^2) \\ h &\equiv (\epsilon/s)g \\ j &\equiv -(L-1)scg \end{aligned} \quad (10)$$

Using these we can rewrite the equation for  $w$

$$w^2 + 2w(h + ju) - (fu^2 + gv^2) = 0 \quad (11)$$

The solution is

$$w = -(h + ju) + [(h + ju)^2 + fu^2 + gv^2]^{1/2} \quad (12)$$

#### 4. Expansion

We now expand Equation 12 in a Taylor series. Since  $u$  and  $v$  are small,  $h$  is the largest term. We obtain after some algebra and regrouping of terms,

$$\begin{aligned} w &= \frac{1}{2h}(fu^2 + gv^2) - \frac{j}{2h^2}(fu^3 + guv^2) + \\ &\quad \frac{1}{8h^3}[f(4j^2 - f)u^4 - g^2v^4 + 2g(2j^2 - f)u^2v^2] + \dots \end{aligned} \quad (13)$$

Conversion to polar coordinates is achieved with the following standard relations

$$\begin{aligned} u^2 &= \frac{(ap/k)^2}{2}(1 + \cos 2\theta) \\ v^2 &= \frac{(ap/k)^2}{2}(1 - \cos 2\theta) \\ u^3 &= \frac{(ap/k)^3}{4}(3\cos\theta + \cos 3\theta) \\ uv^2 &= \frac{(ap/k)^3}{4}(\cos\theta - \cos 3\theta) \\ u^4 &= \frac{(ap/k)^4}{8}(3 + 4\cos 2\theta + \cos 4\theta) \\ v^4 &= \frac{(ap/k)^4}{8}(3 - 4\cos 2\theta + \cos 4\theta) \\ u^2v^2 &= \frac{(ap/k)^4}{8}(1 - \cos 4\theta) \end{aligned} \quad (14)$$

Inserting these relations into Equation 13 and collecting the terms according to Equation 4 yields

$$\begin{aligned}
 \alpha_{20} &= (a/k)^2 \frac{f+g}{2h} \\
 \alpha_{22} &= (a/k)^2 \frac{f-g}{2h} \\
 \alpha_{31} &= -(a/k)^3 \frac{b}{8h^2} (3f+g) \\
 \alpha_{33} &= -(a/k)^3 \frac{j}{8h^2} (f-g) \\
 \alpha_{40} &= (a/k)^4 \frac{1}{64h^3} [3f(4j^2-f) - 3g^2 + 2g(2j^2-f)] \\
 \alpha_{42} &= (a/k)^4 \frac{1}{64h^3} [4f(4j^2-f) + 4g^2] \\
 \alpha_{44} &= (a/k)^4 \frac{1}{64h^3} [f(4j^2-f) - g^2 - 2g(2j^2-f)]
 \end{aligned} \tag{15}$$

Now plugging in the definitions from equations (7b) and (10) yields the final expressions for the coefficients.

$$\begin{aligned}
 \alpha_{20} &= \frac{a^2}{k} \left[ \frac{2 - K\epsilon^2}{4(1 - K\epsilon^2)^{3/2}} \right] \quad (\text{focus}) \\
 \alpha_{22} &= \frac{a^2}{k} \left[ \frac{K\epsilon^2}{4(1 - K\epsilon^2)^{3/2}} \right] \quad (\text{astigmatism}) \\
 \alpha_{31} &= \frac{a^3}{k^2} \left[ \frac{K\epsilon[1 - (K+1)\epsilon^2]^{1/2}(4 - K\epsilon^2)}{8(1 - K\epsilon^2)^3} \right] \quad (\text{coma}) \\
 \alpha_{33} &= \frac{a^3}{k^2} \left[ \frac{K^2\epsilon^3[1 - (K+1)\epsilon^2]^{1/2}}{8(1 - K\epsilon^2)^3} \right] \\
 \alpha_{40} &= \frac{a^4}{k^3} \left[ \frac{8(1+K) - 24K\epsilon^2 + 3K^2\epsilon^4(1-3K) - K^3\epsilon^6(2-K)}{64(1 - K\epsilon^2)^{9/2}} \right] \quad \left( \begin{array}{l} \text{spherical} \\ \text{aberration} \end{array} \right) \\
 \alpha_{42} &= \frac{a^4}{k^3} \left[ \frac{K\epsilon^2[2(1+3K) - (9+7K)K\epsilon^2 + (2+K)K^2\epsilon^4]}{16(1 - K\epsilon^2)^{9/2}} \right] \\
 \alpha_{44} &= \frac{a^4}{k^3} \left[ \frac{K^2\epsilon^4[1 + 5K - K\epsilon^2(6+5K)]}{64(1 - K\epsilon^2)^{9/2}} \right]
 \end{aligned} \tag{16}$$

In general,

$$\alpha_{mn} = o \left( \frac{a^m \epsilon^n}{k^{m-1}} \right)$$

For a sphere,  $K = 0$ , and equation (16) reduces to

$$\begin{aligned}
 \alpha_{20} &= \frac{a^2}{2k} \\
 \alpha_{40} &= \frac{a^4}{8k^3}
 \end{aligned} \tag{17}$$

For a parabola,  $K = -1$ , and equation (16) reduces to

$$\begin{aligned}
 \alpha_{20} &= \frac{a^2}{k} \left[ \frac{2 + \epsilon^2}{4(1 + \epsilon^2)^{3/2}} \right] \\
 \alpha_{22} &= -\frac{a^2}{k} \left[ \frac{\epsilon^2}{4(1 + \epsilon^2)^{3/2}} \right] \\
 \alpha_{31} &= -\frac{a^3}{k^2} \left[ \frac{\epsilon(4 + \epsilon^2)}{8(1 + \epsilon^2)^3} \right] \\
 \alpha_{33} &= \frac{a^3}{k^2} \left[ \frac{\epsilon^3}{8(1 + \epsilon^2)^3} \right] \\
 \alpha_{40} &= \frac{a^4}{k^3} \left[ \frac{3\epsilon^2(8 + 4\epsilon^2 + \epsilon^4)}{64(1 + \epsilon^2)^{9/2}} \right] \\
 \alpha_{42} &= \frac{a^4}{k^3} \left[ \frac{\epsilon^2(4 - 2\epsilon^2 - \epsilon^4)}{16(1 + \epsilon^2)^{9/2}} \right] \\
 \alpha_{44} &= -\frac{a^4}{k^3} \left[ \frac{\epsilon^4(4 - \epsilon^2)}{64(1 + \epsilon^2)^{9/2}} \right]
 \end{aligned} \tag{18}$$

## 5. Verification of the Expansion

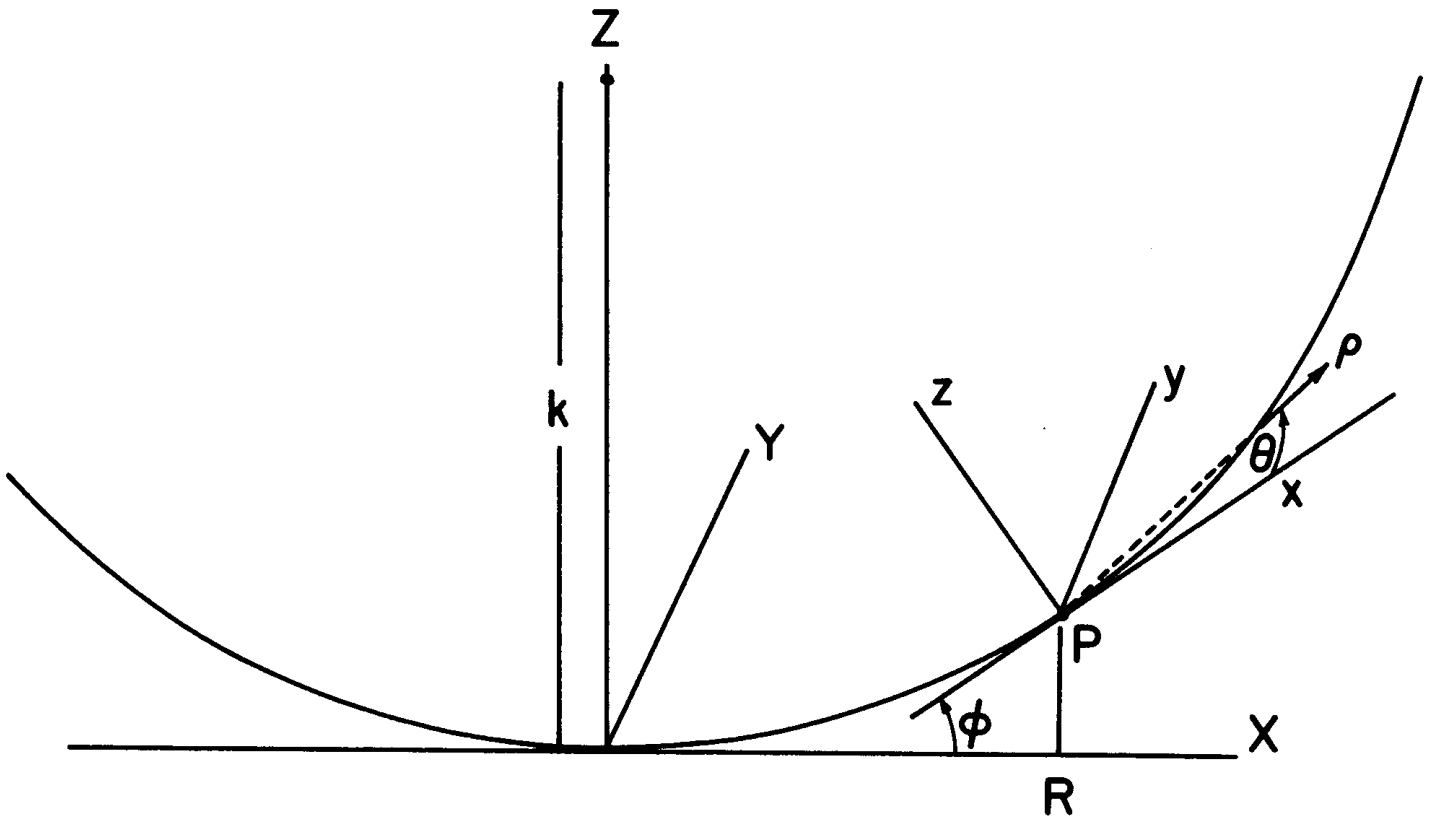
Because the algebraic derivation of the above expressions is rather involved, we felt that an independent check of equation (16) was desirable. This has been done numerically. The procedure is based on the ease of transforming a single point on the conic from one coordinate system to the other, using equation (6). For each of a variety of cases, a grid of points (approx. 700) on the given surface was generated in the parent coordinate system, and each point was transformed to the local coordinate system using equation (6). This set of data points was then fit in a least squares sense with Zernike polynomials. Twenty eight polynomials (through 6th order) were used, and the resulting coefficients were then used to calculate the coefficients defined in equation (4). Conic constants ranging from  $-2$  to  $+1$  were used, along with a variety of off-axis distances. In all cases, the numerical results agreed with equation (16) to the expected accuracy of the numerical computation. Numerical errors were roughly  $10^{-12}$  of the basic surface amplitude. Even the smallest terms were checked to accuracies better than  $10^{-4}$ . Thus the correctness of the algebraic expressions has been confirmed numerically.

## 6. Examples

The hexagonal segments envisioned for the University of California Ten Meter Telescope have a segment radius  $a=0.9\text{m}$ . Thirty-six such segments make up the primary mirror. The  $f/1.75$  primary has  $k=35\text{m}$ . Since the primary will be extremely close to a paraboloid, we will assume for this example that  $K=-1$ . Using equation (18) we calculate the expansion coefficients as a function of the off-axis distance. The results are shown in figure 2. The marks along the abscissa indicate the actual off-axis distances of the segments. As is true for all but the most extreme telescope configurations, astigmatism ( $\alpha_{22}$ ) is the dominant aberration, followed by coma ( $\alpha_{31}$ ). We have also included on the plot the root mean square deviation of the surface from the best fitting sphere. The sensitivity of the coefficients to the conic constant is indicated in figure 3. Here we show the coefficients over a range of  $K$  from  $-2$  to  $+2$ . Note that for all coefficients the variation is smooth. In particular, we note that hyperbolic surfaces with conic constants near  $-1$  are quite similar to parabolas.

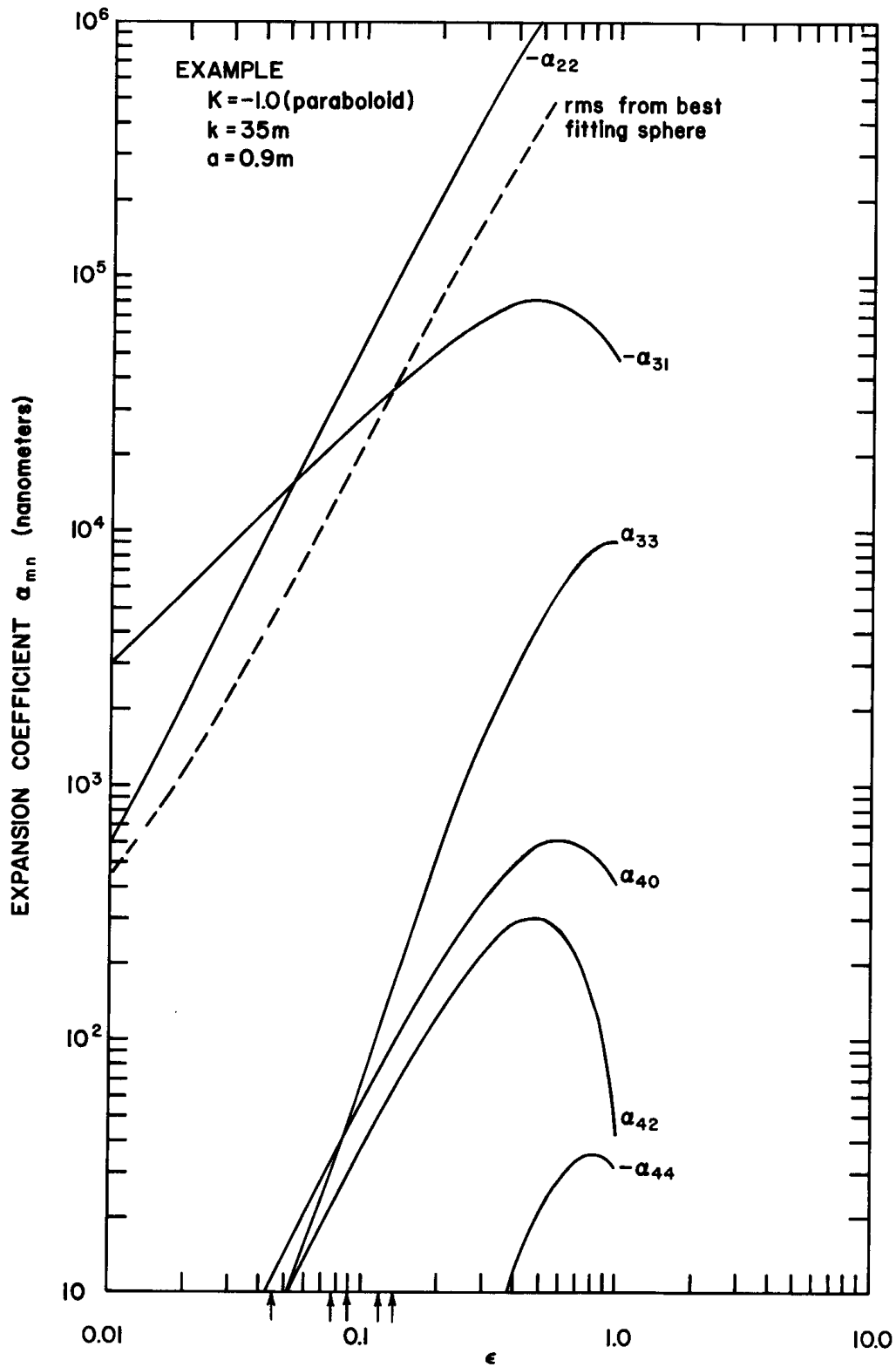
### **Acknowledgements**

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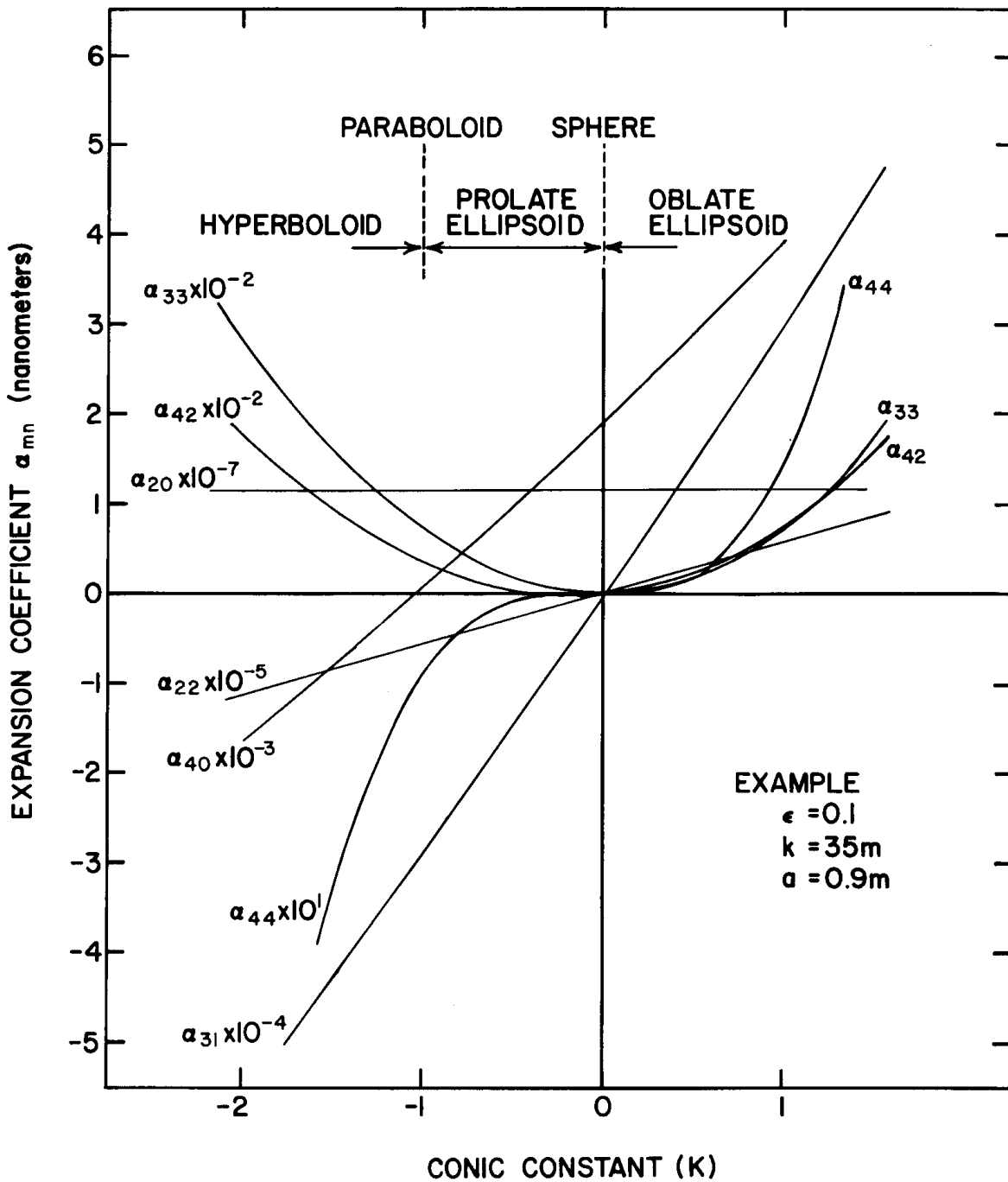
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Fig. 1. Diagram defining global  $(X, Y, Z)$  and local coordinates  $(x, y, z = r, \theta, z)$  of mirror segment on conic.



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Fig. 2. Expansion coefficients describing the surface of an off axis section of a conic as a function of off-axis distance are shown. In this example the conic is a paraboloid, and a segment radius of  $a = 0.9\text{m}$  and a radius of curvature  $k = 35\text{m}$  are assumed. The arrows along the abscissa indicate the off axis distances for the segments of the UC Ten Meter Telescope. The focus term has been omitted. The dashed line gives the root mean square difference between the conic and the best fitting sphere.



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Fig. 3. The variation of the expansion coefficients as a function of the conic constant is shown. For this example we have assumed  $a = 0.9m$ ,  $k = 35m$ , and  $R = 3.5m$ .